

Thm: Let  $T$  act on  $X \hookrightarrow \mathbb{P}(V)$ ,  
 then the functor  $R \mapsto \text{Map}_T(\text{Spec } R, X)$  is representable by a closed subscheme of  $X$ .  
 ↑ trivial  $T$ -action

It is smooth if  $X$  is smooth.

PF: can reduce to affine case using sumihiro, and suffices to assume  $k=k$ , so  $T$  is lin. reductive.

$A$  is  $M$ -graded algebra, let  $I \subset A$  be the ideal gen. by  $\bigoplus_{x \in M \setminus \{0\}} A_x$ .

Then  $Z = \text{Spec}(A/I)$  represents the functor. It is smooth because can show that

$T_x Z = (T_x X)^T$ ; and can lift non-invariant elements of  $m_x/m_x^2$  to elements of  $m_x$  which define  $Z$  as a transverse intersection at  $x$ .

↳ another approach: smooth  $\iff$  fin. type & formally smooth

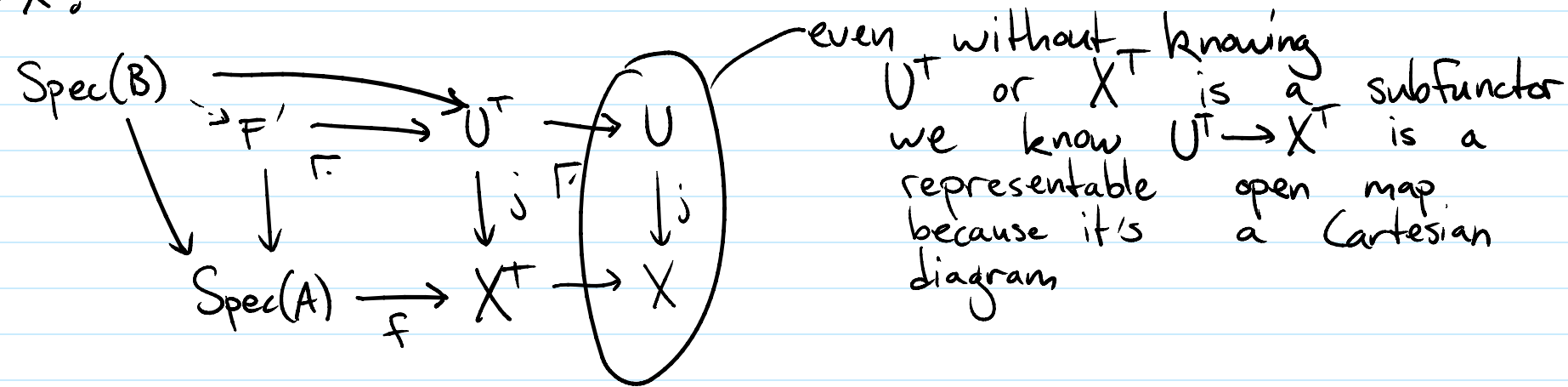
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Aside on technique: map of functors  $F \rightarrow G$  is repr. and open if  $\forall$  maps

$$\begin{array}{ccc} F' & \longrightarrow & F \\ \downarrow & & \downarrow \\ \text{Spec } A & \xrightarrow{g} & G \end{array} \rightsquigarrow F' \text{ is functor corresp. to open subscheme}$$

Lem.: if  $F$  has a representable open cover by schemes, it is a scheme.

In our case,  $G = X^T$  and  $F = U^T$  for an invariant open affine  $U \subset X$ .



Thm (Białynicki-Birula):  $X \hookrightarrow \mathbb{P}(U)$  is  $G_m$ -proj. then the functor

1)  $R \longmapsto \text{Map}_{G_m}(\text{Spec}(R) \times A^1, X)$  is representable by a scheme  $Y$

2) the restriction  $\text{Map}_{G_m}(\text{Spec}(R) \times A^1, X) \xrightarrow{i} \text{Map}(\text{Spec}(R) \times \mathbb{A}^1, X)$  is an embedding,  $\sigma: X \times_{G_m} \rightarrow Y$  closed embedding as well

an embedding  $\sigma: X \hookrightarrow Y$  closed embedding as well  
 3) if  $X$  is smooth, then  $\text{Map}_{\mathbb{G}_m}(\text{Spec}(R) \times A^1, X) \xrightarrow{\pi} \text{Map}_{\mathbb{G}_m}(\text{Spec}(R) \times \{0\}, X)$   
 is a locally trivial bundle of affine spaces

PF (Idea) Similar to last, can reduce to the affine case. In that case  $\text{Spec}(A/I^+)$  represents the functor, where  $I^+$  is the ideal gen. by elements of positive weight.

↳ differences:  $\circ$  open cover is by  $\pi^{-1}$   
 $\circ$  of open cover of  $X^{\mathbb{G}_m}$



Ex:  $\mathbb{P}(V)$  with a linear action of  $\mathbb{G}_m$

→ choose eigenbasis, i.e. coordinates s.t.  $\mathbb{G}_m$ -action is

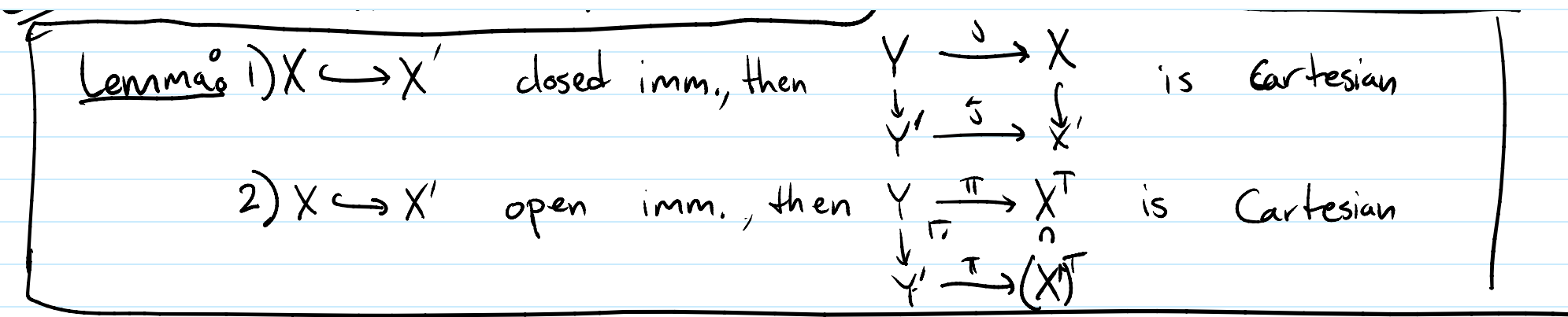
$$[t^{a_0} z_0 : \dots : t^{a_n} z_n] \quad \text{with } a_0 \leq \dots \leq a_n$$

→ then fixed loci are eigenspaces for each  $a$

→ BB strata look like  $[0 : \dots : 0 : 1 : * : * : \dots]$

Exercise → Also strata on general  $X$  are almost restriction of strata in  $\mathbb{P}^n$

Lemma:  $X \hookrightarrow X'$  closed imm. then  $Y \xrightarrow{j} X$  is Cartesian



Ex<sup>o</sup> if  $G$  is split reductive any parabolic is standard meaning there is some IPS  $\lambda: \mathbb{G}_m \rightarrow G$  and a linear embedding such that

$$P_\lambda = G \cap \{ \text{block upper triangular matrices} \}$$

w.r.t. corresponding filtration

If  $G$  not reductive then can still define  $P_\lambda$ , but  $G/P_\lambda$  won't be proper unless  $P_\lambda$  contains unipotent radical of  $G$