Fixed loci and Bialynicki-Birula strata

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<u>Thm:</u> Let T act on $X \hookrightarrow \mathbb{P}(V)$, then the functor $R \longmapsto \mathrm{Map}_T(\mathrm{Spec}\,R, X)$ is representable by a closed subscheme of X

trivial T-action

It is smooth if X is smooth.

<u>Pf:</u> can $\boxed{\text{reduce}}$ to affine case using sumihiro, and suffices to assume $k=\bar{k}$, so T is lin. reductive

A is M-graded algebra, let $I \subset A$ be the ideal gen. by $\bigoplus_{x \in M \setminus \{0\}} A_x$

Then $Z = \mathrm{Spec}(A/I)$ represents the functor. It is smooth because can show that $T_x Z = (T_x X)^T$; and can lift non-invariant elements of m_x/m_x^2 to elements of m_x which define Z as a transverse intersection at x. ∎

↳ another approach: smooth $\Leftrightarrow$ fin. type & formally smooth

<u>Aside on technique:</u> map of functors $F \to G$ is repr.

↳ <u>Aside on technique</u>: map of functors $F \to G$ is repr. and open if $\forall$ maps

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \arrow[d] \\ \operatorname{Spec} A \arrow[r, "g"] & G \end{tikzcd}$$

$\rightsquigarrow$ F' is functor corresp. to open subscheme

Lem: if F has a representable open cover by schemes, it is a scheme.

In our case, $G = X^T$ and $F = U^T$ for an invariant open affine $U \subset X$.

$$\begin{tikzcd} \operatorname{Spec}(B) \arrow[dr, dashed] \arrow[rr, bend left] \arrow[ddr] & & U^T \arrow[r] \arrow[dd, "j"] & U \arrow[dd, "j"] \\ & F' \arrow[ur] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & & \\ & \operatorname{Spec}(A) \arrow[r, "f"] & X^T \arrow[r] \arrow[uur, phantom, "\ulcorner"] & X \end{tikzcd}$$

even without knowing U^T or X^T is a subfunctor we know $U^T \to X^T$ is a representable open map because it's a Cartesian diagram

Thm (Białynicki-Birula): $X \hookrightarrow \mathbb{P}(V)$ is $\mathbb{G}_m$-qproj. then the functor

1) $R \longmapsto \operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \mathbb{A}^1, X)$ is representable by a scheme Y

2) the restriction $\operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \mathbb{A}^1, X) \xrightarrow{i} \operatorname{Map}(\operatorname{Spec}(R) \times \{1\}, X)$ is an embedding, $\sigma : X^{\mathbb{G}_m} \hookrightarrow Y$ closed embedding as well

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3) if X is smooth, then $\mathrm{Map}_{\mathbb{G}_m}(\mathrm{Spec}(R) \times \mathbb{A}^1, X) \xrightarrow{\pi} \mathrm{Map}_{\mathbb{G}_m}(\mathrm{Spec}(R) \times \{0\}, X)$ is a locally trivial bundle of affine spaces

__Pf:__ (Idea) Similar to last, can reduce to the affine case. In that case $\mathrm{Spec}(A/I^+)$ represents the functor, where I^+ is the ideal gen. by elements of positive weight.

↳ differences: open cover is by π^{-1} of open cover of $X^{\mathbb{G}_m}$ ☒

__Ex:__ $\mathbb{P}(V)$ with a linear action of $\mathbb{G}_m$

⇝ choose eigenbasis, i.e. coordinates s.t. $\mathbb{G}_m$-action is $[t^{a_0} z_0 : \cdots : t^{a_n} z_n]$ with $a_0 \leq \cdots \leq a_n$

→ then fixed loci are eigenspaces for each a

→ BB strata look like $[0 : \cdots 0 : 1 : * : * : \cdots]$

→ Also, strata on general X are almost restriction of strata in $\mathbb{P}^n$

Exercise

Lemma: 1) $X \hookrightarrow X'$ closed imm. then $Y \xrightarrow{j} X$ is Cartesian

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$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d] & X \arrow[d, hook] \\ Y' \arrow[r, "\tilde{j}"] & X' \end{tikzcd}$$
is Cartesian

2) $X \hookrightarrow X'$ open imm., then
$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X^T \arrow[d, phantom, "\cap"] \\ Y' \arrow[r, "\pi"] & (X')^T \end{tikzcd}$$
is Cartesian

__Ex:__ if G is split reductive any parabolic is __standard__ meaning there is some 1PS $\lambda : \mathbb{G}_m \to G$ and a linear embedding such that

$$P_\lambda = G \cap \{\text{block upper triangular matrices}\}$$
w.r.t. corresponding filtration

If G not reductive then can still deffine P_λ, but G/P_λ won't be proper unless P_λ contains unipotent radical of G